Pursuit on Graphs

Contents

1. Introduction

1.1. **The game.** The aim of this essay is to give an overview of some topics concerning a game called *Cops and Robbers*, which was introduced independently by Nowakowski and Winkler [3] and Quillot [4]. Given a graph G, the game goes as follows. We have two players, one player controls some number k of *cops* and the other player controls the *robber*. First, the cops occupy some vertices of the graph, where more than one cop may occupy the same vertex. Then the robber, being fully aware of the cops' choices, chooses a vertex for himself [1]. Afterwards the cops and the robber move in alternate rounds, with cops going first. At each step any cop or robber is allowed to move along an edge of G or remain stationary. The cops win if at some time there is a cop at the same vertex as the robber; otherwise, i.e., if the robber can elude the cops indefinitely, the robber wins. The minimum number of cops for which there is a winning strategy, no matter how the robber plays, is called the *cop number* of G, and is denoted by $c(G)$. We note that if initially we place a cop on every vertex of G then the cops will win in the first round, and hence $c(G)$ is well-defined.

1.2. **Examples.**

1.2.1. *The 5-cycle.* Suppose $G = C_5$ the 5-cycle. Label the vertices of G as in the picture.

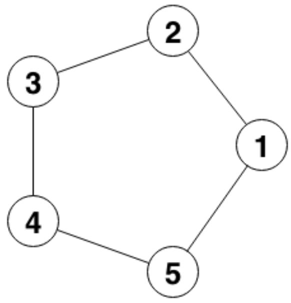

Figure 1. The 5-cycle

Suppose we only use one cop, and place her on vertex 1, say. If the robber spawns in 2 or 5, then he gets caught in the first round. If he spawns in 3 or 4 however, he can always move so that his distance from the cop after his move is 2. Hence he can avoid the cop forever - so one cop is not enough to catch the robber.

Two cops however are clearly enough. Just put both cops on vertex 1, and let one of them move clockwise, the other anticlockwise. The robber will get caught in at most two rounds. So $c(C_5) = 2$. Similarly we conclude that $c(C_n) = 2$ for any $n \geq 4$.

[1] We usually refer to the cops as female and the robber as male as in [6] - some sources do the opposite, e.g. [9], but Prof. Bollobás said (in personal communication) that this may be sexist.

1.2.2. *The path.* If $G = P_n$, the path of length n, then one cop can win easily. Put the cop anywhere, and just keep moving towards the robber. Eventually the robber gets caught, so $c(P_n) = 1$.

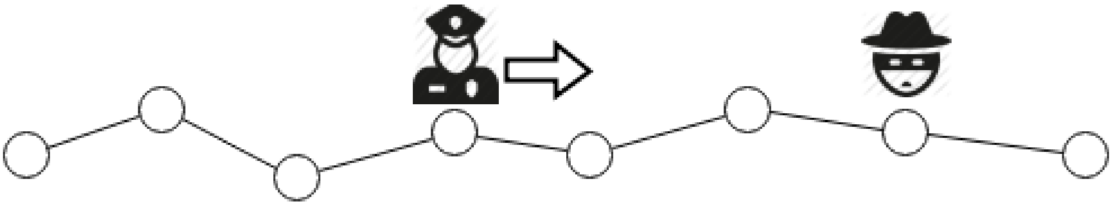

FIGURE 2. One cop is enough on a path

1.2.3. *The complete graph.* If $G = K_n$ the complete graph on n vertices, one cop can win: we put the cop anywhere, and catch the robber in our first move. So $c(K_n) = 1$.

1.2.4. *The tree.* Suppose G is a (finite) tree. We claim that one cop can catch the robber. We put the cop anywhere, and at each step we move towards the robber on the unique shortest path connecting the cop and the robber. After each move of the cop, the number of vertices to which the robber can move without crossing the cop decreases by at least one. Hence after a finite number of steps the robber gets cornered and caught. So $c(G) = 1$.

1.2.5. *General graphs.* Calculating the cop number of general graphs is a NP-hard problem [10]. Similarly as for the chromatic number, we run into difficulties when calculating the cop number even for small graphs - the Petersen graph P has $c(P) = 3$, but proving this already requires some analysis.

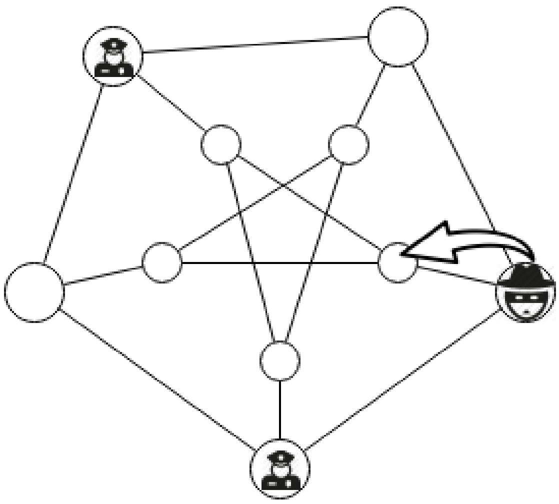

FIGURE 3. A robber can always evade two cops

Throughout the essay, we will assume that G is connected and simple, because deleting multiple edges or loops does not affect the possible moves of the players, and the cop number of a disconnected graph equals the sum of the cop numbers for each component.

2. Lower bounds

2.1. **A first idea.** In this section we want to prove theorems of the form "*If G is a graph of order n that satisfies some properties, then $c(G) \geq f(n)$*". The obvious way to prove such a theorem would be to show that for such a graph, and a small number of cops, the robber has an *escaping strategy.* What does an escaping strategy look like? At each step, the robber has to be able to move to a vertex that has no neighbours occupied by cops. We say a cop *controls* a vertex v, if the cop is on v or the cop is on a neighbour of v. Hence, to show that the robber has an escaping strategy, we must show that the robber can always move to a vertex that is not controlled by a cop. To prove a theorem of the above form, we just need to show that less than $f(n)$ cops cannot control all neighbours of any vertex v. It would be nice to have a bound on the number of neighbours controlled by a single cop.

2.2. **Aigner and Fromme.** How can we make sure that a single cop cannot control many neighbours of a vertex? Suppose first that v is a vertex of G, and the cop is in a vertex c, connected to v. Then the number of neighbours of v that are controlled by the cop equals the number of triangles in the graph that contain both v and c, plus one. So if G contains no triangles, then the cop controls precisely one neighbour of v (namely c).

Now suppose c is not connected to v. If the distance of v and c is at least 3, then the cop controls no neighbours of v. Otherwise, if the distance is 2, then the number of neighbours of v controlled by the cop equals the number of common neighbours of v and c, i.e. the number of 2-paths from c to v. Note that if G contains no 4-cycles, then there is only one such 2-path, so the cop only controls one neighbour. In general, if the number of 4-cycles containing both c and v is k, and the cop controls $m \geq 1$ neighbours of v, then $k = \binom{m}{2}$.

Hence if G has girth at least 5, and v is any vertex with no cops on it, then every cop can control at most one neighbour of v. So if the minimal degree of G is larger than the number of cops, the robber can always move to a non-controlled vertex. This is the theorem of Aigner and Fromme. Below we give a slightly easier proof than the one in their original paper, but the key ideas are exactly the same.

Theorem 2.2.1 (Aigner and Fromme, 1984)**.** [11] *Let G be a graph with minimum degree $\delta(G)$ which contains no 3- or 4-cycles. Then $c(G) \geq \delta(G)$.*

Proof. Given k cops, with $k < \delta(G)$ we show that the robber can always evade them. Suppose the cops initially choose vertices $c_1, c_2 \dots c_k$. Let v be any vertex not occupied by the cops (such vertex exists, as $|G| > \delta(G) > k$). As G has girth at least 5, v has a neighbour u that is not controlled by any cop. Put the robber on u. Whatever strategy the cops use during the game, the robber always has a neighbour that is not controlled by the cops. Hence the robber can always move to an uncontrolled vertex, and hence can evade the cops forever. □

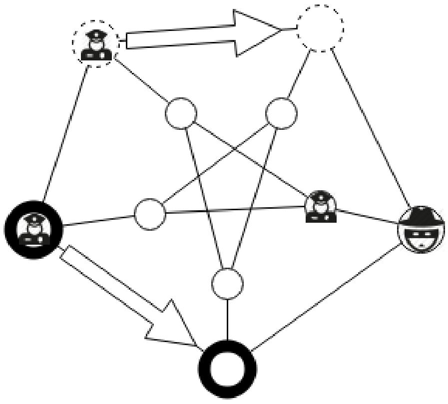

FIGURE 4. Every cop controls one neighbour of the robber

Corollary 2.2.2. *The Petersen graph P has $c(P) = 3$.*

Proof. Theorem 2.2.1 gives $c(P) \geq 3$. Since the diameter of P is 2, the neighbourhood of any vertex is a dominating set. ☐

Corollary 2.2.3. *The Hoffman-Singleton graph H has $c(H) = 7$. The Robertson-Wegner graph has $c(G) \geq 5$. The last Moore graph G, if it exists, has $c(G) = 57$.*

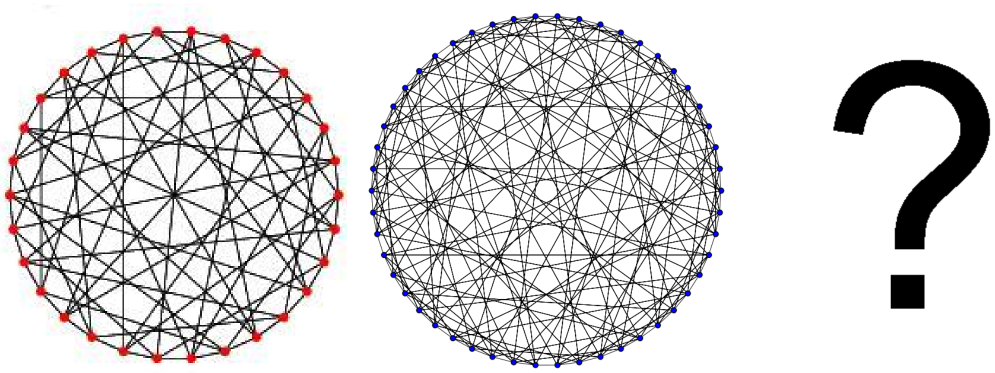

FIGURE 5. The Robertson-Wegner graph, the Hoffman-Singleton graph and the missing Moore graph

Corollary 2.2.4. *Let P be a projective plane of order q. Let G be the bipartite graph formed from P by letting one vertex set be the points of P and the other vertex set be the lines of P, connecting a point and a line in G if they are incident in P. Then $c(G) \geq q + 1 \approx \frac{\sqrt{n}}{2}$.*

(In fact, it is a nice exercise to show that $c(G) = q + 1$ in the above.)

2.3. **Easy generalisations of Aigner and Fromme.** How can we generalize the above theorem? The first thing we note is that we didn't need *all* the vertices to have large degree - it is enough if there is a subgraph H with large minimal degree, where the robber can move so that he evades the cops forever.

Theorem 2.3.1. *Let G be a graph of girth ≥ 5, and let $H \subset G$ have minimum degree $\delta(H)$. Then $c(G) \geq \delta(H)$.*

Proof. The proof is very similar to the proof of 2.2.1. Suppose we have $k < \delta(H)$ cops, initially placed on some vertices of G. Pick a vertex v of H not occupied by any cops. Then v has a neighbour u in H, not controlled by any cops. Put the robber initially on u. Now the robber announces that he will only move on the vertices of H. At every point in the game, the robber has a neighbour in H not controlled by the cops, and hence can evade them forever. □

Given a graph G and an integer $k > 0$, the *k-core* of G is the maximal subgraph H_k with minimum degree at least k. The *degeneracy* of a graph G is the largest k for which the k-core of G exists. Equivalently, the degeneracy of G is the least k for which every subgraph of G has a vertex of degree at most k.

Corollary 2.3.2. *Let G be a graph of girth ≥ 5 and degeneracy k. Then $c(G) \geq k$.*

Proof. Let H be the k-core of G. The statement then follows from 2.3.1. □

What if we didn't insist on G having girth 5, but allowed the graph to contain a few smaller cycles? Indeed, we only needed the cops to be unable to control all the neighbours of any vertex in some subgraph H of G. To make this idea work, given a graph G, define the function $f_G : V(G) \to \mathbb{N}$ as follows. Given a vertex v of G, let $f_G(v)$ be the least integer k such that one can place k cops on the vertices of $G \setminus \{v\}$ such that the cops control all neighbours of v. Similarly, if H is any subgraph of G, define a function $f_G^H : V(H) \to \mathbb{N}$ as follows. Given a vertex v of H, let $f_G^H(v)$ be the least k such that one can place k cops on the vertices of $G \setminus \{v\}$ such that the cops control all neighbours of v in H. Note that $f_G^G \equiv f_G$.

Lemma 2.3.3. *Let G be a graph. Then*

$$c(G) \geq \max_{H \subset G} \left\{ \min_{v \in V(H)} f_G^H(v) \right\}$$

In particular, $c(G) \geq \min\{f_G(v) : v \in V(G)\}$.

[Side note: observe that the right hand side of the above expression is always at most $\sqrt{n}$ (see [26]). Hence this lemma cannot be used to disprove Meyniel's Conjecture (3.1.1 and 3.1.3).]

Proof. Suppose the number k of cops is less than the above expression. Then $\exists H \subset G$ such that for every $v \in V(H)$, we have $k < f_G^H(v)$. After the cops have chosen any initial position they want, pick a vertex v of H not occupied by any cops. Such vertex exists, as for any $v \in V(H)$ we have $f_G^H(v) \leq |H|$. This vertex v has a neighbour u in H that is not controlled by the cops, put the robber on u. Now the robber announces that he will only move on the vertices of H.

He can do this while always evading the cops, as the cops cannot control all neighbours of any vertex in H. □

Corollary 2.3.4. *Let G be a strongly regular graph with parameter set (v, k, λ, μ), that is, a regular graph on v vertices and degree k, such that every two adjacent vertices have λ common neighbours and every two non-adjacent vertices have μ common neighbours. Then $c(G) \geq \min\{\frac{k}{\lambda+1}, \frac{k}{\mu}\}$.*

Proof. Let v be any vertex of G. If a cop is in a neighbour of v then it controls exactly $\lambda + 1$ neighbours of v. Similarly, if a cop is at distance 2 from v, then it controls μ neighbours of v. If the cops cannot control all neighbours of a vertex, then the robber can evade forever. □

Corollary 2.3.5. *The Higman-Sims graph G has $c(G) \geq 4$. The Hall-Janko graph has $c(G) \geq 3$.* □

Now we try to find lower bounds for the above functions. Given a graph G and a vertex v, let $s_3(v)$ be the number of triangles in G containing v, and similarly let $s_4(v)$ be the number of 4-cycles in G containing v, and let $s_4(u, v)$ be the number of 4-cycles in G containing both u and v. Suppose we have k cops available, and want to place them on vertices of $G\backslash\{v\}$ such that together they control all neighbours of v. We will put k_1 cops on neighbours of v, and k_2 cops at distance 2 from v, where $k_1 + k_2 = k$. The k_1 cops on neighbours of v cover in total at most $S_1 = k_1 + s_3(v)$ neighbours of v, and this bound is attained if we didn't put two cops on vertices c_1, c_2 such that together with v they form a triangle, and if all triangles containing v have a cop on it. What can we say about the other k_2 cops at distance 2, how many neighbours of v do they cover in total?

Let these cops be on distinct vertices $c_1, c_2, \ldots, c_{k_2}$. Write $g(l)$ for the number of neighbours of v covered by cop l. Then $\binom{g(l)}{2} = s_4(v, c_l)$, and so

$$g(l) = \frac{1}{2} + \sqrt{\frac{1}{4} + 2s_4(v, c_l)}$$

Note that

$$\sum_{i=1}^{k_2} s_4(v, c_i) \leq s_4(v)$$

because no two of these cops can be on the same 4-cycle with v (that would imply one of the cops is a neighbour of v). So the total number S_2 of neighbours of v covered by these k_2 cops satisfies

$$S_2 \leq \sum_{l=1}^{k_2} g(l) \leq k_2 \left(\frac{1}{2} + \sqrt{\frac{1}{4} + 2\frac{s_4(v)}{k_2}} \right) = \frac{k_2}{2} + \sqrt{\frac{k_2^2}{4} + 2k_2 s_4(v)} \tag{1}$$

So the total number of neighbours of v covered by the k cops is at most

$$S = S_1 + S_2 \leq k_1 + s_3(v) + \frac{k_2}{2} + \sqrt{\frac{k_2^2}{4} + 2k_2 s_4(v)}$$

As $k_1 + k_2 = k$, the maximum occurs at $k_2 = k$, and we get

$$S \leq \frac{k}{2} + s_3(v) + \sqrt{\frac{k^2}{4} + 2ks_4(v)} \leq k + s_3(v) + \sqrt{2ks_4(v)}$$

We note that if a vertex has very few 4-cycles, we can improve inequality (1). Since the values of the $s_4(v, c_i)$ have to be integers, and $\binom{g(l)}{2} = s_4(v, c_l)$, we get $g(l) \leq s_4(v, c_l) + 1$, and hence

$$S \leq k + s_3(v) + s_4(v)$$

Theorem 2.3.6. *Let H be any subgraph of G and $k > 0$ be an integer. Suppose that for all vertices v of H, at least one of the following inequalities hold:*

$$d_H(v) > \frac{k}{2} + s_3(v) + \sqrt{\frac{k^2}{4} + 2ks_4(v)}$$

$$d_H(v) > k + s_3(v) + s_4(v)$$

where $d_H(v)$ denotes the degree of v in H. Then $c(G) > k$.

Proof. Both inequalities imply that for a vertex v in H, k cops cannot cover all neighbours of v in H. So $f_G^H(v) > k$ and hence we are done by 2.3.3. □

2.4. **Graphs of larger girth.** In the previous section we considered graphs of girth 5, and graphs of few triangles and 4-cycles. Can we say something new if instead we insisted on the girth being large? Can we prove a theorem of the form "If G is a graph of order n and girth g then $c(G) \geq f(n, g)$"?

The natural thing to try is to imitate the proof of Aigner and Fromme. There we said that since there are no $3-$ and $4-$cycles, any cop can only cover one neighbour of a given vertex. Now we want to say something more, given that we don't have any $5-$, $6-$, $\ldots$ cycles either. To do that we need to generalise the notion of 'covering' - let's call it *distance covering* for now. We want to say, that "Given a vertex v, any cop can distance cover only one neighbour of v. But there are d neighbours, so $c(G) \geq d$".

But hang on - we proved $c(G) \geq d$ for girth 5 graphs already, we want something larger on the right hand side. A natural thing to consider instead of neighbours, is a neighbourhood of some radius r. How many vertices are there in this neighbourhood? Well, if $r \leq \frac{g-1}{2}$, and every vertex has degree at least d as in the theorem of Aigner and Fromme, then there are at least $S_r = d + d(d-1) + \cdots + d(d-1)^{r-1}$ vertices at distance at most r from a given vertex.

Surely we have generalised all the concepts and now the proof should go through, right? So let's start the proof as in Aigner and Fromme. Suppose we have only a few cops, less than S_r, say. Then every cop can distance cover (whatever that means) only one point in this neighbourhood. So there is a point that is not distance covered, so the robber goes there and escapes forever... But wait! This uncovered point might be quite far from the original vertex. So how can we make sure there is no cop waiting on that vertex by the time the robber gets there? We need to be very careful with the definition of distance covering!

So suppose we have a good definition for distance covering. Since the uncovered point might be at any distance (between 1 and r) from the robber, it makes sense to think in blocks of r

steps. So maybe we should say "a cop distance covers a vertex if it can get there in at most r steps"? There are two problems with our argument now. First, if the uncovered vertex is a neighbour of the robber, and there was a cop right next to the robber initially, then the robber takes 1 step to the safe vertex and waits $r-1$ rounds (because we think in blocks of r rounds) and thus gets caught by the cop. A second problem is that this definition of distance covering will definitely not imply that one cop can only distance cover one vertex at most. Fortunately, there is an easy solution to both flaws - we got our generalisation of neighbours wrong!

What if instead of a closed ball we only considered vertices of distance exactly r from the robber? There are $T_r = d(d-1)^{r-1}$ of those if $r \leq \frac{g-1}{2}$. We say a cop distance covers a vertex if it can get there in at most r steps. Now the crucial part is to say that one cop can only distance cover one vertex. Well, is that true? Certainly not – if a cop is inside the r-ball around the robber then all the vertices whose shortest path to the robber go through the cop are distance covered by him.

The way to get over this, as is the case often with proofs by induction, to prove a slightly stronger statement. We will prove that the robber can always move to a vertex v such that all 'close' cops are in the same direction from v – meaning there is a neighbour u of v such that all shortest paths from these close cops to v go through u. Then we will consider blocks of "t" steps at once, and prove that the robber can move to another such safe vertex in t steps and not get caught on the way. We will call a cop 'close' to the robber if their distance is at most r.

Why is this a good thing to try to prove? Well, there are $(d-1)^t$ vertices at distance t from the robber in v, that are 'away' from u, meaning the robber doesn't have to go through u to get there in t steps. So if the robber moves to one of these vertices x in t steps, then a close cop (in vertex c, say) can't get there in t steps if he is going through v so he has to find another way. But that other way has to be longer than t - otherwise we have found a cycle of length $r+2t$, which, for appropriate choices of r and t will be way less than the girth g.

Now we have a good feeling that this argument will work, we only need to get the details right. Suppose the robber is in v and has a neighbour u such that all shortest paths from v to cops at distance at most r go through u. Consider now the far cops. For an appropriate choice of r we will be able to say that a far cop distance covers at most one vertex at distance t from v. There are at least $(d-1)^t$ vertices to be distance covered, so if there are fewer cops than that, then there is a safe vertex x where the robber can go in t steps. Call its predecessor y, i.e. the last vertex the robber goes before x. If a cop was close to v then its new shortest path to x will have to go through y, otherwise we can find a small cycle. If the cop was very far, further than $r+2t$, then it will still be a far cop when the robber is in x. If the cop was at distance between r and $r+2t$ of v, and now after t steps he got close to x, then there is a path from its present position to x through y of length at most $r+4t$. As long as $2r+4t < g$, its new shortest path will have to go through y as well. So we put $r \approx g/4$ and $t \approx g/8$. This will complete the induction.

This is the point where we realize we didn't actually need the any of distance-covering. We only have to say that if a cop is at distance at least $g/4$ from the robber, then there is at most one vertex at distance $g/8$ from the robber that the cop can also reach in $g/8$ steps, otherwise we would have a smaller cycle. More generally, if for any two vertices x and y we can find a path connecting them of length $< g/2$ then we can be sure that it is the unique shortest $x - y$ path. Hence for the far cops we only need to consider where their shortest path to the robber intersects the set of vertices at distance t from the robber. The details of the proof can be found below.

Theorem 2.4.1. [7] *Let t be an integer, and G be a graph of girth $g \geq 8t - 3$. Suppose all vertices in G have degree greater than d. Then $c(G) \geq d^t$.*

Proof. In any (connected finite) graph, if some number of cops have a winning strategy from some initial starting position then they have a winning strategy from any starting position, even if they announce that they will not move for the first 100 rounds – after that they can move to their winning position and catch the robber.

So wlog we may assume that initially all cops are in a vertex b, and the robber starts in vertex a adjacent to b and it's the robber's turn to move. Assume we have k cops, where $k < d^t$ and we will show that the robber can evade them forever.

Call a vertex v *safe* if it has a neighbour u that satisfies the following: after the cops' move, if a cop's distance to v is at most $2t-2$ then the shortest path (unique, as $2t-2 < g/2$) connecting the cop to v goes through u.

Initially, the pair (a, b) is safe: all cops are in b, hence their shortest paths to a clearly goes through b. We will show that the robber can move in t steps to another safe vertex and not get caught on the way.

Suppose now the robber is in a safe vertex v with neighbour u. Let S be the set of vertices which are at distance t from v, whose shortest path to v doesn't go through u. As t is much less than $g/2$, there are at least d^t such vertices.

We distinguish 3 types of cops. There are the *close* cops which are at distance at most $r = 2t - 2$ from v. There are the *semi-far* cops which are not close, but their distance from v is less than $g/2$. And there are the *far* cops.

Now consider the semi-far cops, i.e. the cops at distance more than r but less than $g/2$ from v, and draw the shortest path from each such cop to v. Note that these shortest paths are unique, i.e. there can not be two shortest paths from such a cop to v. Each path intersects S in exactly one point, hence as the number of cops was less than d^t there is a vertex x in S that doesn't lie on any of these paths. The strategy of the robber is to go straight to x, say vertex y is the one before x. We claim that x is safe with neighbour y.

First we show that the robber didn't get caught on the way to x. Well, he certainly didn't get caught by the semi-far and far cops, as $2t - 2 = r$. Similarly, he didn't get caught by the *close* cops either, since the shortest path from x to a close cop must go through v, and hence there is no shortcut from the cop to x.

Now to show that x is safe. If a cop was far initially, then he will still be at least semi-far, as $g/2 - 2t > r$. So far cops are ok. Now consider the close cops. Suppose a cop was on c and moved to c' now. Then there is a path from c' to x through c, v and y of length at most $r + 2t < g/2$ so this is the unique shortest path connecting c' and x. So close cops are ok too. What about the semi-far cops? Well, if a semi-far cop was in c but got close now, say he moved to c', then the distance of c and v must have been at most $r + 2t$. So we have a path from c' to x going through c, v, and y of length at most $r + 4t$. We also have a path from c' to x of length at most r because the cop got close. If this second path doesn't go through y then we could find a cycle of length at most $2r + 4t < g$. So semi-far cops are ok as well.

So we conclude x is safe and the robber can evade the cops forever. □

By being a bit more careful, one can actually get $c(G) > d^t$ above. Then the above proof for $t = 1$ gives the result of Aigner and Fromme.

We note that in March 2014, Yuqing Lin has found a much simpler proof for the stronger result where $8t - 3$ is replaced by $6t - 1$ in the above:

Proof idea. Show that the robber can maintain a distance of at least t from all cops throughout the game. □

3. Upper Bounds

3.1. **Main conjectures.** The most well-known conjecture in this area is *Meyniel's conjecture*:

Conjecture 3.1.1 (Meyniel's conjecture). *For any connected G of order n, we have $c(G) = O(\sqrt{n})$.*

Some examples of graphs that attain this bound are the incidence graphs of projective planes (see 2.2.4) and the Moore graphs (see 2.2.3). This conjecture remains wide open, in fact no-one even knows how to show the so-called *soft Meyniel's conjecture*:

Conjecture 3.1.2 (Soft Meyniel's conjecture). *There exists an absolute constant $c > 0$ such that for any connected G of order n, we have $c(G) = O(n^{1-c})$.*

We don't even know whether $c(G) = O(n^{0.9999})$! However, there is some evidence that in fact the following should be true (and best possible):

Conjecture 3.1.3. [26] *For any connected G of order n, we have $c(G) \leq \sqrt{n}$.*

We give three arguments from [26] that support conjecture 3.1.3:

(1) A Moore graph on n vertices has $c(G) = \sqrt{n-1}$. These graphs have many extremal properties - so perhaps they have the largest cop number as well.[2]
(2) The bound in 2.3.3 cannot be used to disprove 3.1.3. In fact, the supremum of the right hand side is about $\sqrt{n}$. This is equivalent with saying that the teleporting cop number is always at most $\sqrt{n}$.
(3) Every graph contains at least one vertex where $\sqrt{n}$ cops can trap a robber (i.e. they control all of his neighbours). This is not true if we replace $\sqrt{n}$ by $C\sqrt{n}$ for some $C < 1$.

[2]One might argue that there are only finitely many Moore graphs, hence maybe something even stronger than 3.1.3 is true for large enough n. It is true that the highest we can go with inifnite families is $c(n) \approx \sqrt{n/2}$ at present [22]. However, 2.3.6 shows that if a graph is 'almost-Moore' then it has cop number close to that of a Moore graph - this strongly suggests the existence of an infinite family with $c(n) \approx \sqrt{n}$.

The best current general upper bound is $c(G) = O(n^{1-o(1)})$, which was discovered by Lu and Peng [2] and independently by Scott and Sudakov [1]. But before looking at their proof, let's first enjoy some earlier proofs of weaker results.

3.2. **The first non-trivial upper bound.** In their paper, Aigner and Fromme proved another easy but very useful lemma, that we shall build on.

Lemma 3.2.1 (Aigner and Fromme, 1982)**.** [11] *Let G be a graph, and u, v be any two vertices of G. Let P be a shortest path from u to v. Then a single cop can, after finitely many moves, prevent the robber from entering P - that is, he will get caught if he moves onto P.*

Proof sketch. Let $P = \{u, u_1, u_2, \dots u_{k-1}, u_k (= v)\}$ a shortest path between u and v. For $i = 0, 1 \dots k-1$, let $D_i = \{x \in V(G) : d(u, x) = i\}$ and let $D_k = \{x \in V(G) : d(u, x) \geq k\}$. For any l, if the robber is in D_l then he can only move to D_{l-1} or D_{l+1}. The strategy of the cop is to pretend the robber is on P and to catch his shadow. The cop will move on P such that after a while she will always be in the same D_i as the robber. □

The reason why this lemma is so useful is that if G is a graph, and P is a large shortest path in G, then we can put a cop on it to guard it, and hence $c(G) \leq 1 + c(G - P)$.

It would be nice to show that every graph has a very long shortest path in it, as removing it would decrease the number of vertices the robber can go to by a lot. But unfortunately that is not always the case. For example if G has diameter 2 then every shortest path has length at most 3, so this would give us something like $c(G) \leq n/3$ - but we want a result better than that.

It would be nice to find some other exotic structure $\mathcal{L}$ that can be guarded by a single cop. Then we could prove something like "Every graph has a large shortest path or a large $\mathcal{L}$". So $\mathcal{L}$ should be something that dense graphs have a lot of. The simplest answer is: vertices with large neighbourhoods!

Now our strategy is clear: we want to prove that every graph of order n has a shortest path of order $f(n)$ or a vertex of degree at least $f(n) - 1$, for some function f. If we could prove this then we would have $c(n) \leq 1 + c(n - f(n))$, where $c(n) = \max\{c(G) : |G| = n\}$, and then apply induction.

Lemma 3.2.2. *In any connected G of order n, there is a vertex of degree $f(n)$ or a shortest path of order $f(n)$, where $f(n) = \frac{\log n}{\log \log n}$*

Proof sketch. Suppose there is no vertex of degree at least $f(n)$. We want to say that the graph has large diameter. Pick any vertex v. There are at most $f(n)$ neighbours of v, there are at most $f(n)^2$ vertices at distance 2, etc. So if $n > f(n) + f(n)^2 + \cdots + f(n)^{f(n)-1}$ then the graph has diameter at least $f(n)$. It remains to check an easy calculation. □

Theorem 3.2.3. [7] *For any G of order n, we have $c(G) \leq O\left(n^{\frac{\log \log n}{\log n}}\right)$*

Proof. Let S be a vertex of degree at least $\frac{\log n}{\log \log n}$ and its neighbourhood, or a shortest path of order at least $\frac{\log n}{\log \log n}$. Put a cop to guard S. Now the robber can only move on the vertices of

$G\backslash S$. If $G\backslash S$ is connected, then $c(G) \le 1 + c(G\backslash S)$. If not, then $c(G) \le 1 + \max\{c(G_i)\}$ where the G_i are the connected components of $G\backslash S$.

Let $c(m) = \max\{c(G) : |G| = m, G \text{ is connected}\}$. Then from the above,

$$c(n) \le 1 + c\left(n - \frac{\log n}{\log\log n}\right)$$

Now an induction gives the required result. □

A natural question to ask is, what if instead of vertices of large degree, and shortest paths, we considered some other exotic objects? To prove Meyniel's conjecture, we would want some theorems of the form

- Every graph of order n has a *splork* or order $\sqrt{n}$, and
- Any splork can be guarded by 100 cops

Unfortunately, this approach is unlikely to work. The reason why shortest paths were guardable by one cop is that shortest paths are retracts in graphs. However, Bollobás, Kun and Leader proved that some graphs do not have large retracts. In fact they put a poly-logarithmic upper bound on the size of retracts in some graphs, hence this approach will not even get us to the soft Meyniel's conjecture, unless we manage to characterise the graphs with small retracts, and prove Meyniel for them separately.

Here we briefly mention that Chiniforooshan improved [25] the above bound by removing the $\log\log n$ factor. He did so by introducing new objects called *minimum distance caterpillars* that are always 5-guardable.

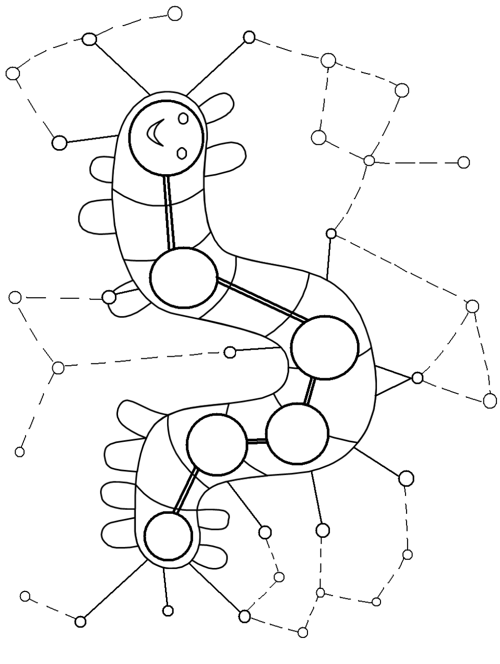

Figure 6. A minimum distance caterpillar

3.3. **An upper bound for graphs of large girth.** We start by making an easy observation. Suppose we manage to initially place the cops on vertices such that every vertex of the graph has a cop in one of its neighbours. Clearly the cops will win, wherever the robber decides to place herself. A *dominating set* of a graph G is a set D of vertices such that every vertex not in D is adjacent to at least one member of D. The *domination number* of a graph G, denoted by $\gamma(G)$ is the number of vertices in a smallest dominating set for G. Hence $c(G) \leq \gamma(G)$.

Now for given $k \geq 1$, consider a subset D_k of vertices of G such that for every vertex v not in D_k, there is an element $u \in D$ such that their distance is at most k. Call such a set a *k-dominating set*. If D_k is such that for every vertex not in D_k there is an element in D_k at distance *precisely* k then call D_k a *strictly k-dominating set*. Write $\gamma_k(G)$ for the size of the smallest strictly k-dominating set. Note that $\gamma_1(G) = \gamma(G)$. Frankl's key observation in the following theorem is that if the girth of the graph is sufficiently large, then by putting the cops in D_k the robber cannot escape and hence $c(G) \leq \gamma_k(G)$.

Lemma 3.3.1. *Given a graph G and an integer $k > 0$. Suppose $c(G) > k$. Then the robber can evade k cops without ever stepping on a vertex of degree 1.*

Proof. Given our original strategy for a robber R, our new strategy for R' is as follows. The new robber R' does exactly the same as R, but whenever R would step onto a vertex of degree 1 according to the original strategy, R' passes and stays where he was. He will continue moving like R when they meet again, i.e. when R has left the leaf vertex. It is clear that if the cops can catch R' then they can catch R in at most one more round. □

Corollary 3.3.2. *Let G be a graph, and let H be its 2-core, i.e. H is obtained from G by repeatedly removing vertices of degree 1 from G. Then if the robber can evade k cops, he can do so without ever leaving H.*

Theorem 3.3.3. [7] *Suppose that the girth of G is at least $4h - 1$. Then $c(G) \leq \gamma_h(G)$.*

Proof. Let D be a strictly h-dominating set of size $\gamma_h(G)$, and place the cops on the vertices of D initially. Suppose the robber decides to start on vertex x. Let S be the set of vertices at distance precisely $h-1$ from x. For each $y \in S$ write $c(y)$ for a cop at distance precisely h from y. We note that if $y, y' \in S$ then $c(y) = c(y')$ is possible. The strategy of the cops is to move in a straight line towards x (as the girth is at least $4h - 1$ and their distance to x is at most $2h - 1$ there is a unique shortest path from each $c(y)$ to x) and stop as soon as they are on the shortest path connecting x and the robber. Our claim is that the robber is caught in at most $2h - 1$ steps, i.e. after all the cops have stopped. Wlog the robber doesn't leave the $2-$core of G.

Suppose after $2h - 1$ steps the robber is in z. If $d(x, z) \geq h - 1$ then the shortest path connecting z to x meets S in some vertex y. If $d(x, z) < h - 1$ then as x, z are in the 2-core of G there is a $y \in S$ such that the shortest path connecting y to x goes through z. So we can pick a $y \in S$ such that either y lies on the shortest path between x and z, or z lies on the shortest path between x and y. We claim $c(y)$ caught the robber in the first $2h - 1$ steps. Let the initial position of $c(y)$ be a.

Suppose after $2h-1$ steps $c(y)$ is in vertex u and didn't catch the robber on the way. If z is on the shortest path between u and x then $c(y)$ stopped at some point prior to reaching x. So at some point u was on the shortest path between the robber and x. But then there was no way for the robber to get between u and x without crossing the cop standing on u, as the girth is $4h-1$. So z is not on the shortest path between u and x.

The cop stopped in u either because $u = x$ or because u lied on the shortest path between the robber and x. In either case, since the girth is at least $4h-1$, the robber must have gone through u at some point in the first $2h-1$ steps – and since he didn't get caught, he went through u before $c(y)$ got there.

Let P_1 be the (unique) shortest path between y and x, and P_2 the one between the initial position of $c(y)$, i.e. vertex a, and x. Let v be the first vertex where these two paths meet, i.e. the vertex in their intersection that is the furthest away from x.Then the shortest path between a and y must go through v – otherwise we would have a cycle of length at most $h+h+(h-1)$. Then $h = d(a,y) = d(y,v)+d(v,a)$ and $h-1 = d(y,x) = d(y,v)+d(v,x)$. So $d(v,x)+1 = d(v,a)$. So the distance of the robber and the vertex v was just one less than the distance of the cop and v. Since the cop goes first, if both run straight to v then the robber gets caught. So the vertex u, i.e. where the cop stopped cannot be between v and a, so u lies on the shortest path connecting v and x.

Now $c(y)$ stopped in u because it was on the path between x and the robber. Where was the robber at this point, i.e. when the cop arrived at u? He wasn't on the path between x and y because then he got caught already. If z lies on the path $x-y$ then the robber can't get to z without going through u. Similarly, is y lies on the path $x-z$ then the shortest path from the robber's current position to z goes through u because of the girth constraint, so the robber can't get to z. These were the only two possibilities due to our choice of y, and hence we are done.

□

Recall that a *hypergraph* H is a set V of vertices together with a set E of edges, where each edge is an arbitrary subset of V. The *degree* of a vertex x is the number of edges that contain x. A *vertex cover* for H is a subset V that meet all edges in E. We write $\tau(H)$ for the size of the smallest vertex cover. A natural generalisation of a vertex cover is the *fractional vertex cover* $\tau^\star(H)$ to be the $\min\{\Sigma_{v\in V} f(v) \mid f : V \to \mathbb{R}^+,\ \forall e \in E(H) :\ \Sigma_{v\in e} f(v) \geq 1\}$. So instead of picking specific vertices, we assign positive weights to some vertices such that the sum of weights in each edge is at least 1.

Theorem 3.3.4 (Lovász (1975), Stein)**.** *If H is a hypergraph of maximum degree d, then $\tau(G) \leq (1+\log d)\tau^\star(G)$.*

Proof idea. Use the greedy algorithm. At each step pick a vertex that covers the most number of new edges. □

Given a graph G, write $N_h(x)$ for the set of vertices at distance precisely h from x. Let $n_h(G) = \min\{|N_h(x)| : x \in V(G)\}$ and $m_h(G) = \max\{|N_h(x)| : x \in V(G)\}$

Corollary 3.3.5. [7] *If G is a graph of girth at least $4h-1$ then $c(G) \leq (n/n_h)(1+\log m_h)$.*

Proof. Let H be a hypergraph on vertex set $V(G)$ with edge set $\{N_h(v) : v \in V(G)\}$. By assigning weight $1/n_h$ to each vertex, we get a fractional vertex cover, hence $\tau^\star(H) \leq n/n_h$. By theorem 3.3.4, we get $\gamma_h(G) = \tau(H) \leq (n/n_h)(1 + \log m_h)$. The result follows from theorem 3.3.3. □

3.4. **A general upper bound.** The best known upper bound is due Lu and Peng [2]. Shortly after, Sudakov and Scott found [1] a similar but shorter proof of the same result, independently from Lu and Peng. Later Frieze, Krivelevich and Loh [13] gave a reformulation of the proof of Sudakov and Scott, using expanders. Here we present only the proof of Sudakov and Scott.

Theorem 3.4.1. *If G is any graph of order n then $c(G) \leq n \cdot 2^{-(1+o(1))\sqrt{\log n}}$*

The main structure of the proof is as follows: if the graph has large diameter then we can find a large shortest path, put a cop on it and delete it. Otherwise, if the graph has small diameter, we will proceed with a random argument.

We will choose some subsets $C_1, C_2, \dots C_t$ of vertices uniformly at random with some probability p for some t, and put a cop on each element in them (if a vertex is present in more sets then we put more cops on it). We will choose p small enough so that the sum of the sizes of the sets isn't too large, but large enough so that our proposed strategy for the cops will work. These sets will be our *squads* of cops.

The robber chooses some vertex v for herself. Now our strategy is as follows. We will define some subsets $A_1, A_2, \dots A_t$ of vertices that have rather few neighbours. The task of squad 1 is to make sure that after 1 round, the robber has to go to A_1, i.e. the cops in squad 1 will occupy all other neighbours of v. Having done so, squad 1 has accomplished its task and they will stay where they are forever. Meanwhile, squad 2 makes sure that after 2 rounds, the robber is cornered into a vertex of A_2. Similarly, the task of squad k is to make sure that after 2^{k-1} rounds, the robber can only be in A_{k-1}. We finish the proof by proving that one of the A_i-s has to be empty, and hence the robber gets caught.

How should we choose the A_i-s? We will define them inductively. Having defined A_i, we consider the closed ball $B(A_i, 2^{i-1})$, and pick a subset of this ball that has rather few vertices at distance $\leq 2^{i-1}$ and call this A_{i+1}. The remaining vertices in this ball have a rather high degree, so we will use Hall's theorem to find a matching from elements of squad i to these vertices.

The difficulty of this proof lies in the details. Even after knowing the main ideas of the proof, filling in the details and making all the estimates work is not easy. The complete proof, a very technical one indeed, is presented below.

We start by dealing with the low diameter case (the hard part!). First, a technical lemma.

Lemma 3.4.2 (Chernoff bound). *Suppose $p \in [0, 1]$, and let $X_1, \dots X_n$ be mutually independent with $P(X_i = 1 - p) = p$ and $P(X_i = -p) = 1 - p$. Let $X = X_1 + \dots + X_n$. Then for any $a > 0$, we have*

$$P(X > a) < e^{-2a^2/n}$$

Proof. See theorem A.1.4 in [12]. □

Now we turn onto the main proof of the low diameter case.

Lemma 3.4.3. *Let G be a graph of order $n \geq 2^{30}$ and diameter $D \leq 2^{\sqrt{\log n}}/\log^3 n$. Then $c(G) \leq n(\log^3 n)2^{-\sqrt{\log n}}$*

Proof. Set $t = \sqrt{\log n} - 3\log\log n$. Note that $2^t = 2^{\sqrt{\log n}}/\log^3 n \geq D$. Let $p = (\log^2 n)2^{-\sqrt{\log n}}$. For each i in $\{1, 2, \ldots t{+}1\}$, choose a subset C_i of vertices uniformly at random with probability p, i.e. for each vertex v and each i, the vertex v has probability p of being in C_i, independently of the other C_j-s. Then the $|C_i|$-s are binomially distributed with expectation $\mu = n(\log^2 n)2^{-\sqrt{\log n}}$. By lemma 3.4.2, the probability that $|C_i| > 2\mu$ is at most $e^{-2\mu^2/n} < e^{-\mu/3} < n^{-2}$. The probability that none of the C_i-s has more than 2μ vertices is at least $(1-n^{-2})^t > 1 - t/n^2 > 1 - \log n/n^2 > 0.9$. So with probability at least 0.9, the sum of the sizes of these sets is at most $2\mu t < n(log^3 n)2^{-\sqrt{-\log n}}$. This will give us our final bound.

To proceed with the proof we need a simple claim. For a subset A of the vertices of G and an integer i, write $B(A, i)$ for the closed ball of radius i around A, i.e. all vertices which can be reached from some vertex in A by a path of length at most i.

Claim: *The following statement holds with probability at least* 0.9*: for every set $A \subset V(G)$ of size at most $n2^{-\sqrt{\log n}}$, every $i \leq t$ such that $|B(A, 2^i)| \geq 2^{\sqrt{\log n}}|A|$, and every j, we have*

$$|B(A, 2^i) \cap C_j| \geq |A|$$

Proof of claim: Let $|A| = a$. Fix A, i, j, and try to estimate the number X of points from C_j in $B(A, 2^i)$. This value is binomially distributed with expectation $\mu = p|B(A, 2^i)| \geq a\log^2 n$. What is the probability that it is less than a? By the Chernoff bound,

$$P(X < a) < e^{-\frac{(a-\mu)^2}{2\mu}} = e^{a - \frac{a^2+\mu^2}{2\mu}} \leq e^{a - \frac{a^2}{2\mu} - a\log^2 n/2} \leq e^{a(1-\log^2 n/2)} \leq e^{-a(\log^2 n)/3}$$

The number of sets of size a is $\binom{n}{a}$, the number of possible choices for i, j is at most $t^2 \leq \log n$, and

$$\log n \sum \binom{n}{a} e^{-a\log^2 n/3} \leq \log n \sum \frac{n^a}{a^a} e^{a(1-\log^2 n/3)} \leq \log n \sum e^{a(1+\log n - \log^2 n/3)} \leq$$

$$\leq n\log n \; e^{2\log n - \log^2 n/3} \leq 10n^4 e^{-\log^2 n/3} \leq 100n^{4-\log n/3} \leq 0.1$$

This proves our claim. □

By our claim, and the result just before that, we can pick our C_i to satisfy the assertion of the claim and such that the sum of their sizes is at most $n(log^3 n)2^{-\sqrt{-\log n}}$. Now for each vertex u put a cop on u for each C_i containing it. Thus we have placed at most $n(log^3 n)2^{-\sqrt{-\log n}}$ on the graph. Remains to show that these cops can catch the robber.

Suppose the robber starts in vertex v. Note first, that $d(v) \leq 2^{\sqrt{\log n}}$ - otherwise, by our claim with $A = \{v\}$, there is a cop in his neighbourhood and he loses in the first round.

Now let A_1 be the largest subset of $B(v, 1)$ such that $|B(A_1, 1)| < 2^{\sqrt{\log n}}|A_1|$, and let $D_1 = B(v, 1) - A_1$. This ensures that if $M \subset B(v, 1)\backslash A_1$ then $|B(M, 1)|$ is large. Indeed, if we had some $M \subset D_1$ with $|B(M, 1)| < 2^{\sqrt{\log n}}|M|$ then we can add M to A_1 to find a larger set, contradicting the maximality of A_1. By our claim, $B(D_1, 1)$ contains at least $|D_1|$ members of C_1.

We want the cops in C_1, i.e. Squad 1, to occupy the set D_1 in their first round. For this to be possible, we need to find a matching from cops in squad 1 to vertices in D_1 such that each cop is adjacent to his assigned vertex. To do this, we want to use Hall's theorem to find a complete matching. But above we showed that for every $M \subset D_1$, we have $|B(M, 1)| \geq 2^{\sqrt{\log n}}|M|$ and hence by our claim $|B(M, 1) \cap C_1| \geq |M|$. So by Hall's theorem we have a perfect matching, and hence Squad 1 can occupy D_1 in their first round.

Now we proceed to define the strategies of the other squads inductively, in a similar way as we did for the first one. Suppose we have defined the strategies for the first i squads, and we have sets A_i, D_i. We want to define A_{i+1}, D_{i+1} such that Squad $i+1$ can occupy D_{i+1} in 2^i steps. Consider the ball $B(A_i, 2^{i-1})$, and let A_{i+1} be the largest subset of this ball such that $|B(A_{i+1}, 2^i)| < 2^{\sqrt{\log n}}|A_{i+1}|$. Let $D_{i+1} = B(A_i, 2^{i-1})\backslash A_{i+1}$. To prove that the cops can occupy D_{i+1}, use Hall's theorem as above. For every $M \subset D_{i+1}$, we have $|B(M, 2^i)| \geq 2^{\sqrt{\log n}}|M|$ and hence by our claim, $|B(M, 2^i) \cap C_{i+1}| \geq |M|$. So by Hall's theorem we have a matching from C_{i+1} to D_{i+1} and hence Squad $i+1$ can occupy D_{i+1} in at most 2^i rounds.

We have shown that we can move the cops in such a way that for each i with $1 \leq i \leq t$, Squad i occupies the set D_i after at most 2^{i-1} rounds. How can the robber evade the cops? After his first round, he must be in $B(v, 1)$ – but the set $D_1 \subset B(v, 1)$ is already occupied by Squad 1. So after his first round he must be in $B(v, 1)\backslash A_1 = D_1$. By induction, suppose after 2^{i-1} rounds he is in A_i. After 2^{i-1} more rounds, he is somewhere in $B(A_i, 2^{i-1})$. But by then, 2^i rounds have passed in total, so Squad $i+1$ has occupied the set D_{i+1}. So the robber has to be in $B(A_{i+1})\backslash D_{i+1} = A_{i+1}$. Hence if the robber manages to evade the cops, then by doing so he has visited each of the A_i-s at least once.

We claim that this is impossible. Indeed, since the diameter of G is small, we must have that $A_t + 1$ is empty and hence the robber could not have possibly visited that set. Note that we chose the A_i-s to have small neighbourhoods. In particular, we have

$$|A_{i+1}| \leq |B(A_i, 2^{i-1})| \leq 2^{\sqrt{\log n}}|A_i|$$

Hence, taking $A_0 = \{v\}$, we get

$$|A_{t+1}| \leq \left(2^{\sqrt{\log n}}\right)^{t+1} = 2^{\log n - 3\log\log n\sqrt{\log n} + \sqrt{\log n}} \leq n2^{-2\sqrt{\log n}}$$

But we also have that

$$|B(A_{t+1}, 2^t)| \leq 2^{\sqrt{\log n}}|A_{t+1}| \leq n2^{-\sqrt{\log n}} < n$$

Since $2^t \geq D$ as remarked in the very first line of this proof, this is a contradiction. Hence A_{t+1} must be empty and hence the robber gets caught in at most 2^t steps. □

It remains to put all the pieces together to make the final induction work.

Proof of Theorem 3.4.1: Let G be a graph on n vertices. We shall prove by induction that

$$c(G) \leq f(n) = 2n\,(\log n)^3\,2^{-\sqrt{\log n}}$$

Note that this trivially holds if $2\,(\log n)^3\,2^{-\sqrt{\log n}} \geq 1$, i.e. if $\log n < 930$, so we may assume $\log n \geq 930$. If G has diameter at most $2^{\sqrt{\log n}}/\log^3 n$ then we are done by Lemma 3.4.3. Otherwise, we have a shortest path P of order at least $D = 2^{\sqrt{\log n}}/\log^3 n$. Put a cop on P to guard this path. Hence we have $c(G) \leq 1 + c(G-P)$. By induction,

$$c(G-P) \leq f(n-D) \leq 2(n-D)\,(\log n)^3\,2^{-\sqrt{\log n - D}} \leq 2n\,(\log n)^3\,2^{-\sqrt{\log n - D}} - 2$$

Since $\log n \geq 930$, we have $D \leq 2^{-900}n$. So

$$\sqrt{\log n - D} \geq \sqrt{\log n - 2D/n} \geq \sqrt{\log n} - 2D/(n \log n)$$

For $x \in (0,1)$ we have $1 + x \geq 2^x$, and hence $2^{2D/(n\log n)} \leq 1 + 2D/(n \log n)$. Substituting back, we get

$$c(G-P) \leq 2n(\log n)^3 2^{-\sqrt{\log n}}\left(1 + \frac{2D}{n\log n}\right) - 2 = f(n) + \frac{4}{\sqrt{\log n}} - 2 \leq f(n) - 1$$

So $c(G) \leq 1 + c(G-P) \leq 1 + f(n) - 1 = f(n)$ and hence we are done.

□

4. Random Graphs

We want to investigate the behaviour of the cop number in random graphs $G(n,p)$ where $p = p(n)$ may depend on n. We shall prove that for constant p we only need $C \log n$ many cops, and that $G(n,p)$ cannot be used to find a counterexample for Meyniel's conjecture.

4.1. **Constant p.**
Consider $G(n,p)$ where p is a constant that does not depend on n. We will show that Meyniel's conjecture holds in this case (and in fact $C \log n$ cops are enough).

First note that if $\gamma(G)$ is the domination number of G, then $c(G) \leq \gamma(G)$. Dreyer proved [14] the following theorem in his doctoral thesis:

Theorem 4.1.1. *Let $p \in (0,1)$ be fixed, and define*

$$\mathbb{L}n = \log_{\frac{1}{1-p}} n$$

Then for every real $\epsilon > 0$, a.a.s.

$$(1-\epsilon)\mathbb{L}n \leq \gamma(G(n,p)) \leq (1+\epsilon)\mathbb{L}n$$

Shortly after, Wieland and Godbole determined [15] the domination number of random graphs where $p = p(n)$ is a function of n:

Theorem 4.1.2. *A.a.s. $\gamma(G(n,p))$ equals one of two values:*

$$\lfloor \mathbb{L}n - (L)\left((\mathbb{L}n)(\log n) + 1\right) \rfloor$$

or

$$\lfloor \mathbb{L}n - (L)\left((\mathbb{L}n)(\log n) + 2\right) \rfloor$$

Later Bonato, Hahn and Wang gave a proof of the following theorem in [16]:

Theorem 4.1.3. *Let $p \in (0,1)$ be fixed, and $\epsilon > 0$. Then a.a.s.*

$$(1-\epsilon)\mathbb{L}n \leq c(G(n,p)) \leq (1+\epsilon)\mathbb{L}n$$

In particular,

$$c(G(n,p)) = \Theta(\log n)$$

Proof idea. The upper bound follows from 4.1.1. We only need to establish the lower bound. We shall use the ideas from Aigner and Fromme's theorem in section 2 – if k cops cannot be placed to cover all neighbours of any vertex, then the robber can always escape forever. By theorem 2.3.3, we only need to prove that a.a.s. $f_G(v) > \lfloor (1-\epsilon)\mathbb{L}n \rfloor$ for all vertices v of $G(n,p)$. □

4.2. **Variable** p. For dense random graphs, Bonato, Pralat and Wang proved [17] the following:

Theorem 4.2.1. *If $d = np = n^{\alpha+o(1)}$, where $1/2 < \alpha \leq 1$, then a.a.s.*

$$c(G(n,p)) = \Theta(\log n/p) = n^{1-\alpha+o(1)}$$

To get a better understanding of the behaviour of $c(G(n,p))$, we define a function $f:(0,1)\to\mathbb{R}$ by

$$f(x) = \frac{\log \bar{c}(G(n,n^{x-1}))}{\log n}$$

Where $\bar{c}(G(n,n^{x-1}))$ is the median cop number of $G(n,p)$. Then the following straight line depicts the conclusion of 4.2.1:

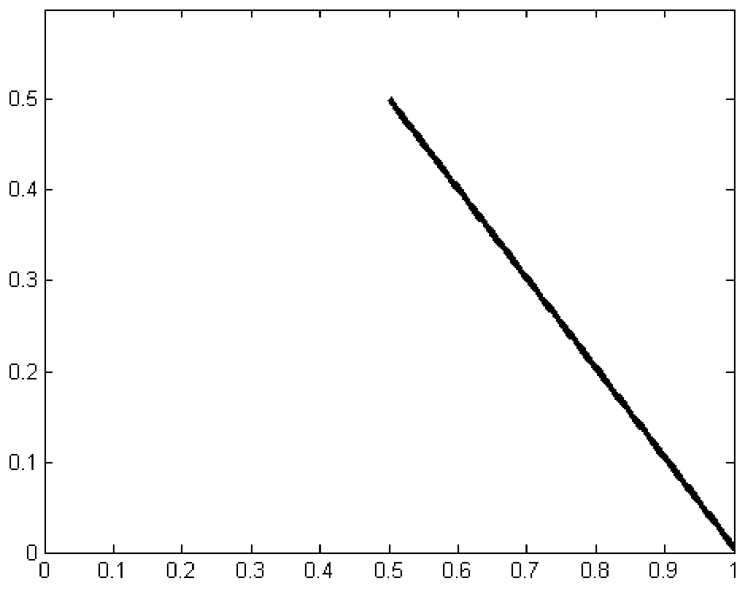

FIGURE 7. The graph of f, so far

Bollobás, Kun and Leader considered sparse random graphs and proved the following bounds [18] in 2008:

Theorem 4.2.2. *If $p(n) \geq 2.1 \log n/n$, then a.a.s.*

$$\frac{1}{(np)^2} n^{\frac{1}{2}\frac{\log\log(np)-9}{\log\log(np)}} \leq c(G(n,p)) \leq 160000\sqrt{n}\log n$$

Hence they proved Meyniel's conjecture for random graphs, up to a logarithmic factor of n.

The following figure shows what we know so far:

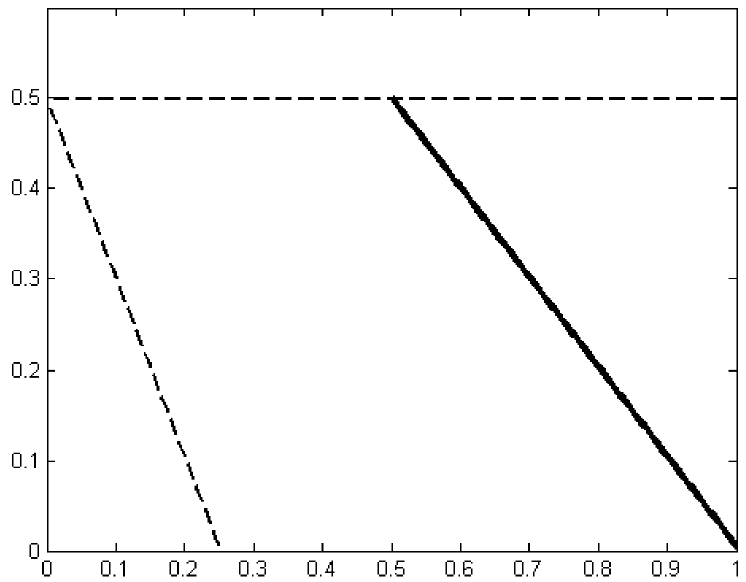

FIGURE 8. The graph of f, so far

The dashed lines denotes the lower and upper bounds.

The above two results show that if $np = n^{o(1)}$ or $np = n^{1/2+o(1)}$ then a.a.s. $c(G(n,p)) = n^{1/2+o(1)}$, hence it would be natural to assume that in between, i.e. for $0 < \alpha < 1/2$, we have that $c(G(n,p))$ is close to $\sqrt{n}$. It turns out that the actual behaviour of our function f is more complicated.

4.3. **The Zigzag Theorem.** Luczak and Pralat showed [19] that the behaviour of the function f follows a zig-zag shape. Their main theorem is as follows:

Theorem 4.3.1. *Let $0 < \alpha < 1$ and $d = d(n) = np = n^{\alpha+o(1)}$.*

(i) *If $\frac{1}{2j+1} < \alpha < \frac{1}{2j}$ for some $j \geq 1$, then a.a.s.*

$$c(G(n,p)) = \Theta(d^j)$$

(ii) *If* $\frac{1}{2j} < \alpha < \frac{1}{2j-1}$ *for some* $j \geq 1$*, then a.a.s.*

$$\Omega\left(\frac{n}{d^j}\right) = c(G(n,p)) = O(\frac{n}{d^j}\log n)$$

Note that 4.3.1 implies the following about the function f:

$$f(x) = \begin{cases} \alpha j, & \text{if } \frac{1}{2j+1} < \alpha < \frac{1}{2j} \text{ for some } j \geq 1, \\ 1 - \alpha j, & \text{if } \frac{1}{2j} < \alpha < \frac{1}{2j-1} \text{ for some } j \geq 1 \end{cases}$$

The case $\alpha = 1/k$ is not covered in their paper for technical reasons. They write: “Nonetheless, one can expect that, up to a factor of $\log^{O(1)} n$, our result extends naturally to the case $np = n^{1/k+o(1)}$ as well”. The following picture shows the graph of the function f (picture taken from [19])

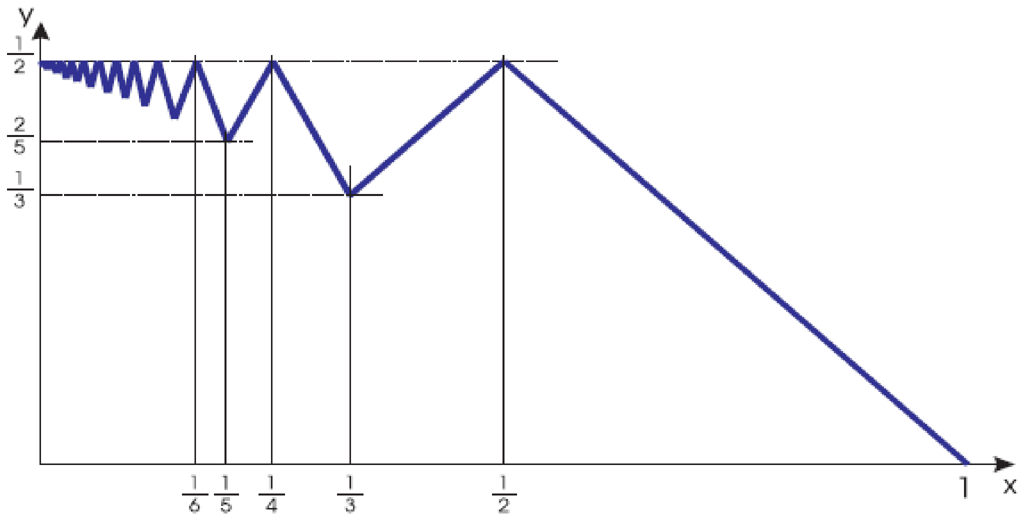

FIGURE 9. The zigzag shape of the function f

The proof of 4.3.1 has two parts. The lower bound has a very clever idea in it, but is rather technical. We start by establishing the upper bound, that has a straightforward proof, but is still a bit technical.

The idea for the upper bound is as follows. Place the cops on vertices at random. Then, if the robber is at v, say, the cops will occupy all vertices at distance exactly j from v, and they will do so in at most $j+1$ rounds. That means that the robber is surrounded in $N_j(v)$ – then the cops will tighten the circle around him until he gets caught.

We start with a lower bound on the sizes of neighbourhoods in $G(n,p)$:

Lemma 4.3.2. *Let $0 < \alpha < 1$, and $d = np = n^{\alpha+o(1)}$. Then a.a.s. for any $1 \le i \le 1/(2\alpha)$ and vertices $v_1, \dots v_k$ of $G(n,p)$, we have*

$$\left| \bigcup_{j=1}^{k} N_{i+1}(v_j) \right| \ge 0.5 \min\{k(0.1d)^{i+1}, n\}$$

Proof sketch. Pick k vertices and generate their $(i+1)$-neighbourhoods one after the other, always disregarding vertices we have already discovered. We may assume we haven't discovered $n/2$ vertices yet. Then by the Chernoff bound (3.4.2), the probability that a vertex gives us less that $(0.1d)^{i+1}$ new vertices is less than n^{-4}. Thus the probability that at least half of the k vertices have less than $(0.1d)^{i+1}$ vertices in their $(i+1)$-neighbourhood is $O(n^{-1})$. □

Proof of the upper bound. (i) Assume $n^{1/(2j+1)} \le d \le n^{1/(2j)}$ and let $\gamma = \lceil n \log n / d^{2j+1} \rceil$. We will prove that a.a.s. $c(G(n,p)) = O(d^j \gamma)$ which is easy to see to imply our upper bound. We begin with placing $\beta n = 5000(10d)^j$ cops on the vertices of the graph at random, and show that these cops can catch the robber. Say the robber chooses v.

We want to assign to each vertex u at distance exactly j from v a unique cop in at distance at most $j+1$ from u. To prove this, we will use Hall's marriage theorem. Consider the bipartite graph, where one vertex set consists of all vertices at distance exactly j from v, and the other vertex set being the vertices occupied by the cops (appearing with the corresponding multiplicity). A vertex and a cop are joined by an edge if their distance is at most $j+1$. We want to show that there is a matching saturating the first vertex set.

Fix a subset S of the vertices at distance precisely j from v, i.e. $S \subset N_j(v) \setminus N_{j-1}(v)$, say of size $|S| = k$. We want to estimate the number of cops in the $(j+1)$-neighbourhood of S. By Lemma 4.3.2, the area of this neighbourhood is at least $0.5 \min\{k(0.1d)^{j+1}, n\}$. Let k_0 be the largest k such that $k(0.1d)^{j+1} \le n$. Hence if $k \le k_0$, we conclude the lower bound on the neighbourhood of S:

$$|N_{j+1}(S)| \ge k(0.1d)^{j+1}$$

Since we placed the cops uniformly at random, the expected number X of cops in this neighbourhood satisfies $E(X) \ge k\beta(0.1d)^{j+1} \ge 50k \log n$.

Recall that we want to show that there are at least k cops in this neighbourhood. The Chernoff bound comes to our help again: The number of cops in this neighbourhood is binomially distributed, hence the probability that there are less than k cops in this neighbourhood is at most $\exp(-4k \log n)$. The sum of these probabilities over all S of size at most k_0 is

$$\sum_{k=1}^{k_0} \binom{|N_j(v)|}{k} \exp(-4k \log n) \le \sum_{k=1}^{n} n^k \exp(-4k \log n) = \sum_{k=1}^{n} n^{-3k} = O(n^{-2}) \tag{2}$$

So a.a.s. the condition in Hall's theorem holds for all subsets of size at most k_0. What about the subsets of size greater than k_0? We can just do the same thing as before. By the Chernoff bound, $|N_j(v)| \le 2d^j$ holds a.a.s., hence if $k_0 < k$ then the number of cops in $N_{j+1}(S)$ is at least $\beta n/4 \ge 50d^j > |N_j(v)|$ by Chernoff, with probability at least $1 - \exp(-4d^j)$. As before, the sum of these probabilities over all S of size at least k_0 is

$$\sum_{k=k_0+1}^{n} \binom{|N_j(v)|}{k} \exp(-4d^j) \leq 2d^j 2^{2d^j} \exp(-4d^j) = O(n^{-2}) \tag{3}$$

So Hall's criterion holds a.a.s., and hence the cops can occupy all vertices at distance precisely j from v before the robber can escape $N_j(v)$.

What now? The cops have trapped the robber in $N_j(v)$, now they have to catch him. The cops' strategy is now to tighten the loop in each step - that is, in the next round they will occupy all vertices at distance precisely $j-1$ from v, then all vertices at distance precisely $j-2$, and so on until the robber gets caught. To do this, we need to find a matching from $N_{i+1}(v)\backslash N_i(v)$ onto $N_i(v)\backslash N_{i-1}(v)$, for each $i < j$. It is an easy exercise to show that this matching exists a.a.s. for all $i < j$, hence the cops can tighten the loop around the robber and win the game.

(ii) Assume now that $n^{1/(2j+2)} \leq d \leq n^{1/(2j+1)}$. We can do exactly the same as above to show that $\beta n = 5000 n \log n/(0.1d)^{j+1}$ cops are enough. □

What is the difference between the cases (i) and (ii) above? In the strategy we gave for the cops, only the cops at distance at most $2j+1$ from v are active. In (i), this means almost all cops, while in (ii) this means only a fraction of the cops. This little odd/even argument seems completely unimportant at first sight, but this is in fact what gives the plot this nice zigzag shape. So it seems that this immediate greedy pursuit strategy we gave for the cops is in some sense the best possible, and this parity problem is not possible to overcome with some cleverer cop strategy.

Now let's turn to proving the lower bound. We will assume that the number of cops is less than the given value, and prove that the robber may escape forever. The key idea is as follows: suppose the robber is in some vertex v. We will give a score to all his neighbours, depending on how dangerous it is to go to that neighbour. A 0-dangerous vertex will be the worst of all - they have a cop on them. A 1-dangerous vertex doesn't have a cop, but has a cop in one of its neighbours. A k-dangerous vertex has c^k cops in its k-neighbourhood, for some suitably chosen c. A vertex that is not dangerous for any k will be called *safe*. We want to show that the robber can always move to a safe vertex, and hence win the game. A slight technical problem with the above is, that we might deal with stupid cops who all start from the same vertex, and always just move towards the robber. This way they will never catch him, but our argument would fail since all neighbours of the robber get (essentially) the same dangerousness score. To fix this, we will temporarily delete the most dangerous neighbour from the graph when assigning the scores. We will see how this overcomes this problem.

Note that this idea of 'always moving to a safe vertex' is essentially the same as with Frankl's proof about the cop number of graphs of large girth. Only the definition of 'safe' has changed, because the setting is different now. Hence the following seems like a really efficient general strategy for proving strong lower bounds on the cop number:

Strategy 4.3.3 (Strategy for proving a lower bound on the cop number of some fixed graph G)**.**

(1) Define the notion of a *safe vertex*

(2) Assume we have less cops than the lower bound that we wish to prove, and show that there exists an initial configuration where the robber is on a safe vertex.

(3) Prove that the robber can always move along safe vertices, starting from the above initial configuration

We will demonstrate the effectiveness of the above recipe by proving the lower bounds for the Zigzag Theorem:

Proof of the lower bound. Assume first that $\frac{1}{2j+1} < \alpha < \frac{1}{2j}$, and let $c = c(j, \alpha) = \frac{3}{1-2j\alpha}$. We will show that a.a.s.

$$c(G(n,p)) \geq \left(\frac{d}{3cj}\right)^j$$

and the lower bound will follow.

For vertices u, v, w, write $C_i^u(w)$ (or $C_i^{u,v}(w)$) for the number of cops at distance at most i from w in $G\backslash\{u\}$ (or $G\backslash\{u, v\}$, respectively). Hence if $u \neq v$ then $C_0^u(v) = 1$ iff there is a cop on v, and $C_1^{u,v}(w) > 0$ iff there is a cop on w, or in a neighbour of w, that is not u or v.

Suppose it is the robber's move. We call a vertex v *safe* if it is not occupied by a cop, and it has a neighbour x such that $C_{2i-1}^x(v), C_{2i}^x(v) \leq \left(\frac{d}{3cj}\right)^i$ for all $i = 1, 2, \ldots, j$. We call x a *deadly neighbour* of v.

Assume that we have less than $\left(\frac{d}{3cj}\right)^j$ cops. If the cops can catch the robber then they can do so from any initial starting position. Pick a vertex w and put all the cops there. Since $\alpha < \frac{1}{2j}$, a.a.s. there is a vertex v at distance more than $2j$ from w. Put the robber on v. Then even after the cops' first move, the vertex v will be safe. Hence it remains to show that whenever the robber is on a safe vertex then he can always move to a vertex that will be safe after the cops' move.

Suppose the robber is on v, with deadly neighbour x. We will assign a score to all neighbours of v. We say that for some $r \geq 0$ a neighbour y of v is *r-dangerous* if

(1) $C_r^{x,v}(y) > 0$ is $r = 0, 1$;

(2) $C_r^{x,v}(y) > \left(\frac{d}{3cj}\right)^i$ if $r = 2i$ or $r = 2i + 1$.

How many r-dangerous neighbours does v have? Let's start with $r = 0$: the number of 0-dangerous neighbours of v is the number of neighbours of v occupied by cops. But since v was safe, $C_1^x(v) \leq \frac{d}{3cj}$ so the number of 0-dangerous neighbours of v is at most $\frac{d}{3cj}$. Write $\mathrm{dang}(r)$ for the number of r-dangerous neighbours of v, not counting x. Hence $\mathrm{dang}(0) \leq \frac{d}{3cj}$, and similarly we can show $\mathrm{dang}(1) \leq \frac{d}{3cj}$. What about larger values of r?

Note that a.a.s. for any two vertices a, b, if $b \in N_i(a)$ for some i with $2 \leq i < 1/\alpha$, then a and b are joined by fewer than $2/(1 - i\alpha)$ paths of length i. (This follows from Markov's inequality.) Hence if in our case w is at distance r from v, then there are less than c neighbours of v which are at distance $r - 1$ from w. We will now compute $C_r^x(v)$ in two different ways. Assume first that $r = 2i$.

On the one hand, v has $\text{dang}(r-1)$ neighbours which are $(r-1)$-dangerous, and each such neighbour has at least $\left(\frac{d}{3cj}\right)^{i-1}$ cops at distance at most $r-1$. Each cop was counted at most c times, hence

$$\left(\frac{d}{3cj}\right)^{i-1} \text{dang}(2i-1) \leq c \cdot C_{2i}^{x}(v)$$

On the other hand we know that v is safe, so

$$C_{2i}^{x}(v) \leq \left(\frac{d}{3cj}\right)^{i}$$

and hence

$$\text{dang}(2i-1) \leq \frac{d}{3j}$$

.

Similarly we can show that if $r = 2i+1$ we get $\text{dang}(2i) \leq \frac{d}{3j}$. Hence we get that for all $r = 0, 1, \dots 2j-1$, we have $\text{dang}(r) \leq \frac{d}{3j}$.

Hence at most $2d/3$ neighbours of v are r-dangerous for some $0 \leq r \leq 2j-1$. So there are about $d/3$ neighbours which seem safe - does it matter which one the robber goes to? Indeed, it does! It could be that x is overflowing with thousands of cops. If the robber moves to a vertex y which is close to x in $G \backslash \{v\}$ then there is no chance for y to be a safe vertex in the next round! So we want - in fact, *need* - to prove for our approach to work, that v has a non-dangerous neighbour y which is at distance at least $2j$ from x in $G \backslash \{v\}$. If we can move to such a y then we would be done. Indeed, the fact that y is non-dangerous and far from x implies that y, with v being its deadly neighbour, is a safe vertex after the cops' move.

To prove that such a y exists, we will show that if $i < 1/\alpha$ then a.a.s. each edge of $G(n,p)$ is contained in at most ϵd cycles of length at most $i+2$. Applying this claim to the edge $x - v$ will imply the existence of such a y.

Let's start by picking two vertices r_1, r_2 that are connected by an edge, and let's try to count how many (r_1, r_2)-paths there are in the graph of fixed length l, for some $2 \leq l \leq i+1$. Denote the number of such paths by $X_j^{r_1,r_2}(p)$. If we can show that the number of such paths is a.a.s. at most $\epsilon d/(2i)$ for all $l \leq i+1$ then we are good. We can easily write down the expectation of this random variable:

$$\text{E}\left(X_j^{r_1,r_2}(p)\right) = \binom{n-2}{j-1}(j-1)!p^j < (np)^j/n \leq d\left(d^i/n\right) < \frac{\epsilon d}{4i}$$

We want to bound the probability that $X_j^{r_1,r_2} > \frac{\epsilon d}{2i}$ above. One possible way to do this is by using Vu's inequality ([20], Cor. 2.6.). We first choose a $p\prime > p$ such that we have an equality above, that is $\text{E}\left(X_j^{r_1,r_2}(p\prime)\right) = \frac{\epsilon d}{4i}$ and then Vu's inequality gives

$$\text{P}\left(X_j^{r_1,r_2}(p\prime) > \epsilon d/(2i)\right) \leq \text{P}\left(X_j^{r_1,r_2}(p\prime) > 2\text{E}X_j^{r_1,r_2}(p\prime)\right) \leq \exp\left(-a(\text{E}X_j^{r_1,r_2}(p\prime))^{1/(j+1)}\right)$$

Since $\frac{\epsilon d}{4i} \geq \log^{2(j+1)} n$, we get

$$\text{P}\left(X_j^{r_1,r_2}(p) > \epsilon d/(2i)\right) \leq \exp(-a\log^2 n) = o(n^{-2})$$

Hence a.a.s. such a y exists and our proof of the Zigzag Theorem is complete.

□

5. Cops and Fast Robber

A natural generalisation of the usual game of Cops and Robbers is where we allow the robber to move more than one edge in his round. This variant was first considered in [21]. The natural question is: what is the corresponding Meyniel's conjecture for this game? There is no chance of proving the best possible bound (since we can't even prove the original Meyniel's conjecture), but it is worth trying to find constructions of graphs of high cop number for this game. In this section we will present the best known constructions, due to Alon and Mehrabian [9].

5.1. Finding the correct bound.

Assume the robber has speed s, so that he can move s edges in his round. Should we allow the robber to pass through cops while moving, as long as he finishes on an empty vertex? It turns out that if we do allow this, then the cop number can be as large as $\Omega(n)$ (see [23]). So let's assume the robber is not allowed to move through cops in his round.

Our strategy is clear. Find some family of graphs G_n, and prove that they have large cop number. To prove that a graph has a high cop number we can use our usual strategy that seems to always work well - see Strategy 4.3.3. Our best bet for the G_n-s are some type of combinatorial designs - they often have very high cop number, and often this isn't completely impossible to prove. So how should we begin our proof?

We suggest two ways of finding the correct bound. The first, probably more natural way, is to open a big book of combinatorial designs, try to find a family that seems to have large cop number, then prove that they indeed do have large cop number, and hope that we found the asymptotically best possible lower bound. This worked well for the usual Cops and Robbers, where the incidence graph of projective planes was an easy design to find and analyze.

However for this game this is unlikely to work, because it is much harder to smell out the optimal designs. We are much better off by doing the proof first, and then find a graph for which our proof applies! This might seem like an absurd idea at first, but we will see that this works, while sniffing out the correct construction without any clues is equivalent to winning the lottery.

So let's jump straight into the proof. Our best bet is to follow the steps in Strategy 4.3.3. In this case, the main idea is in the definition of a *safe* vertex. Once we guessed the correct definition, writing out the rest of the meta-proof (since we don't really know what we are trying to prove) is routine. Once we finish our meta-proof, it is easy to transform it into a proof of a statement like 'If a graph has these properties then $c(G)$ is at least something' - and once we got that we just have to pray that there exists a graph with those properties and that we can find it.

Theorem 5.1.1 (Meta-theorem)**.** *Let G be a graph satisfying certain properties. Then the number of cops required to catch a robber of speed s is at least C. The properties and the bound will be specified during the proof - we'll see what we need.*

Proof. The notation we use is similar to the one in the proof of the Zigzag Theorem. We write $N_k(u)$ for the set of vertices at distance precisely k from vertex u. If $k = 1$ we sometimes just write $N(u)$. As usual, we say a cop *controls* a vertex if the cop is on the vertex or in a neighbour.

A cop controls a path if the cop controls a vertex of the path. The cops control a path if there is a cop controlling the path.

Now comes the important definition: a vertex r is *safe* if there is a subset $X \subset N_s(r)$ of size M such that for all $x \in X$, all shortest $r-x$-paths are uncontrolled. Here M is a suitable number that we will choose later for our proof to work.

Suppose we have less than C cops. We need to show that there is a safe starting position for our robber. Wlog we will put all cops on the same vertex, really far away from the robber so they don't interfere with our game. Note that even if the graph has diameter 2, we can just attach a long path to it and put the cops on its end vertex - this would probably not change our bounds asymptotically, hence we need not worry about the cops for now. So a safe starting position exists as long as there is a vertex v with $|N_s(v)| \geq M$.

So remains to show that the robber can move from a safe vertex to a safe vertex. Suppose the robber is in r, and (since r is safe) there is a set $X \subset N_s(r)$ of size M such that for all $x \in X$ all shortest $r - x$-paths are uncontrolled. Let A be the set of vertices which lie on a shortest path between r and some $x \in X$. Now the cops move to new positions. Since r was safe, there is no cop in A right now (since the cops didn't control any vertex in A before their move). Now the robber has to decide where to go. Since there is no cop in A, the robber can move to any vertex in X. So all we need to show is that there is a safe vertex in X.

How can we find a safe vertex in X? First of all, a safe vertex is not controlled by a cop. So we need to be able to show that the cops don't control all vertices in X. We can bound the number of vertices controlled by the cops by the sum of the degrees of the vertices occupied by the cops - but this approach simply never works, so we are advised to find something better than that. Suppose a cop is on vertex u and controls some $x \in X$. Then $d(u, r) \in \{s - 1, s, s + 1\}$. Which one is it from the three possibilities? If $d(u, r) = s - 1$ then $u \in A$ which is not possible since there are no cops in A. Can we exclude another one of the two remaining possibilities? Sure - if we assume G is bipartite then $d(u, r) \neq s$. So let's assume G is bipartite. Then $d(u, r) = s + 1$, and hence x lies on a shortest $u - r$-path. So assume the graph has the property that if two vertices are at distance $s+1$ then there are at most m shortest $u-r$-paths, where m is a suitable constant we shall specify later. If we assume this property then any cop can control at most m vertices in X.

This is a good start - but what we really have to prove is that there exists an $x \in X$ with lots of uncontrolled *escaping pairs*. What do we mean by that? If S is any set of vertices, then let $N_k^S(u)$ denote the set of vertices v at distance precisely k from u such that all shortest $u - v$ paths avoid the set $S \cap N(u)$. By an escaping pair we mean a pair (x, y) of vertices with $x \in X$ and $y \in N_s^A(x)$. We call x the *head* and y the *tail* of the escaping pair. We say an escaping pair (x, y) is *uncontrolled* if all shortest $x - y$ paths are uncontrolled. So we just need to prove that there is an $x \in X$ which is the head of at least M uncontrolled escaping pairs, since then x is safe and the robber can move there in his round.

If (x, y) is an escaping pair, we call every shortest path between them an *escaping path*. A natural strategy at this point is to try to get estimates on the number of escaping paths, and the number of escaping paths controlled by cops, and then hopefully we can deduce something from that. So let's do that. Pick a vertex v - how many escaping paths contain v? Let $u_1 u_2 \dots u_{s+1}$ be an escaping path with $u_1 \in X$, $u_2 \notin A$, and assume $v = u_i$. First assume $i \neq 1$. Then if d is the maximum degree of the graph, there are at most d choices for $u_2, \dots, u_{i-1}, u_{i+1}, \dots, u_{s+1}$. As we showed above, once we have chosen u_2 there are at most m choices for u_1. So if $v \notin X$ then there are s choices for its position in an escaping path, hence v is on at most $d^{s-1}ms$ escaping paths. If $v \in X$ then there are at most d choices for all other vertices, hence v is the first vertex of at most d^s escaping paths, hence v is on at most $d^s + d^{s-1}ms$ escaping paths.

How many escaping paths do the cops control? We have shown before that a cop controls at most m vertices in X, giving at most $m(d^s + d^{s-1}ms)$ controlled paths. Through the other vertices he can control at most $d^{s-1}ms$ paths. He controls at most $d+1$ vertices, hence he controls at most $m(d^s + d^{s-1}ms) + (d+1-m)d^{s-1}ms \leq 3msd^s$ escaping paths. Hence c cops can control at most $3msd^s c$ escaping paths. Controlling an escaping path decreases the number of uncontrolled escaping pairs by at most 2, hence the number of controlled escaping pairs is at most $6msd^s c$.

Assume for simplicity that every vertex $x \in X$ is the head of at least $2M$ escaping pairs. If there is no $x \in X$ with at least M uncontrolled escaping pairs then every $x \in X$ is the head of at least M controlled escaping pairs. Since $|X| = M$, this gives at least M^2 controlled escaping pairs. Assuming $M^2 > 6msd^s c$ we established the existence of a safe vertex in X and the robber may escape forever. □

Going through the above proof and collecting the conditions we dropped on the way, what we really proved is this:

Theorem 5.1.2. *Let s, d, m be positive integers and q be a positive real such that $qd^s/2$ is an integer larger than m. Let G be a regular bipartite graph of degree d and diameter larger than s with the following properties:*

(1) *For every two vertices u, v of G of at most $s+1$, there are at most m distinct shortest $u-v$-paths.*
(2) *For every vertex u of G and every subset A of size at most m, we have $|N_s^A(u)| \geq qd^s$.*

Then, assuming the robber has speed s, the cop number of G is at least $q^2 d^s / 24ms$.

Proof. Set $M = qd^s/2$ in the above proof. A safe starting position exists since the diameter of G is larger than s. Since for any $x \in X$ we have $|A \cap N(x)| \leq m$, the rest of the above proof goes through. □

So now we have a theorem that could be used to find graphs of high cop number. All we need to do is find some graphs which have the properties described in Theorem 5.1.2. When thinking about regular bipartite graphs of high diameter, the following design comes to mind:

5.2. **The construction.** Let k, s be positive integers, and $d = 2^k$. Let $x_1, x_2, \dots, x_d$ be all the elements of $GF(2^k)$, written as column vectors of length k over $\mathbb{Z}_2$. Let H be the following $1 + k(s+1)$ by d matrix over the field $\mathbb{Z}_2$:

$$H = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_d \\ x_1^3 & x_2^3 & \cdots & x_d^3 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{2s+1} & x_2^{2s+1} & \cdots & x_d^{2s+1} \end{pmatrix}$$

Let S be the set of columns of H, and let G be the graph with vertex set $\mathbb{Z}_2^{1+k(s+1)}$ and with u, v adjacent if $u - v \in S$. This seems to be a good guess – G is clearly d-regular, since $|S| = d$, and bipartite, since two vertices with the same first coordinate cannot be connected. By fixing d and increasing s we are increasing the order of G while not changing the degrees, hence we can make sure the graph has large diameter. Let's hope the other properties work out as well!

First of all, it is not obvious why G is connected. For that, we need to prove that every element of $\mathbb{Z}_2^{1+k(s+1)}$ can be written as a linear combination of vectors in S. That is, we need to show that the rows of H are independent over $\mathbb{Z}_2$. Some coding knowledge comes in handy - the rows $2, 3, \ldots, 1 + k(s + 1)$ generate the dual of a BCH code, and hence all rows are independent (see [24]).

We need to have a control over the number of paths connecting two close vertices. We can do this if d is large compared to s. In fact, we can show that for every two vertices u, v at distance at most $s + 1$ there are at most $(s + 1)!$ distinct shortest $u - v$-paths. To see this, note that if u, v have distance m, then every path connecting them corresponds to a unique ordered representation of the form

$$u - v = s_1 + s_2 + \cdots + s_m,$$

where $s_i \in S$ for all i. Moreover, in such a representation all s_i-s are distinct, since otherwise we could delete them and get a shorter representation. Suppose we have another representation

$$u - v = s'_1 + s'_2 + \ldots s'_m,$$

then since any $(2s+2)$ elements of S are independent (see [12], Lemma 16.2.2), we conclude that the s'_i-s are just a permutation of the s_i-s. Hence the number of $u - v$-paths is just $m!$.

From the linear independence of any $(2s+2)$ columns of S it follows that G has high diameter. Indeed, let $e_1, \ldots, e_{s+1}$ be any distinct elements from S, and let x be their sum. Then the distance between 0 and x is at least $s + 1$.

How about proving that $N_s^A(u)$ is large if $|A|$ is bounded? Suppose $|A| \leq (s + 1)!$, and $A \subset N(u)$. Let $B = \{x \in S : u + x \notin A\}$. Then $|B| \geq d - |A|$. Pick any s-subset of B, say $a_1, a_2, \ldots, a_s$. Then $u + a_1 + \cdots + a_s \in N_s^A(u)$, and since any $2s + 2$ elements of B are independent, any s-subset gives a different element. Hence if $d \geq 2(s + 1)!$ we get

$$|N_s^A(u)| \geq \binom{d - |A|}{s} \geq \left(\frac{d - |A|}{s}\right)^s \geq \frac{d^s}{(2s)^s}$$

.

We know enough now, let's finish this off!

Theorem 5.2.1. *Let s be the speed of the robber. For every n there exists a connected graph of order n with $c(G) = \Omega(n^{s/(s+1)})$*

Proof. First pick k_0 large enough such that $d = 2^{k_0} \geq 2(s+1)!$ and $d^s > 4(s+1)!(2s)^s$. We will only consider the values of n larger than $2^{1+k_0(s+1)}$. Let k be the largest integer such that $2^{1+k(s+1)} \leq n$, and let $n_0 = 2^{1+k(s+1)}$. (We will construct a graph of order n_0, and then add some 'useless' vertices to get a graph of order n. This will not change our asymptotic conclusions since $n < 2^{s+1}n_0$ and so $n = \Theta(n_0)$.)

Let G be the graph described above, with parameters k, s. Now all we need to check is that G satisfies the conditions in 5.1.2. Let $m = (s+1)!$, and define q by $qd^s = 2\lfloor \frac{d^s}{2(2s)^s} \rfloor$. Then G is a d-regular bipartite graph of diameter larger than s, it has $n_0 = O(d^{s+1})$ vertices, between any two vertices at distance at most $s+1$ there are at most m distinct shortest paths, and for any u and any set B of size at most m we have $|N_s^B(u)| \geq (d/2s)^s \geq qd^s$. The way we defined q we know that $q/2$ is an integer, and $q/2 \geq \frac{d^s}{4(2s)^s} > m$ where the last inequality follows from our lower bounds on k_0 and hence d. Hence all conditions are satisfied. We conclude that $c(G) = \Omega(d^s) = \Omega\left(n_0^{s/(s+1)}\right) = \Omega\left(n^{s/(s+1)}\right)$. Now take a path with $n - n_0$ vertices and join it to any vertex of G to obtain G'. Then $|G'|$ has order n and $c(G) = c(G')$.

□

6. Conclusion

We are still very far from solving Meyniel's conjecture, stating that $c(n) \leq C\sqrt{n}$ for all graphs. We know that the conjecture holds for random graphs [19], diameter two graphs [2, 26], planar graphs [11] and several other graph classes (see [6]). There is some evidence that in fact $c(n) \leq \sqrt{n}$ should be true, i.e. that the correct value of C in the above should be one [26]. However, the best general upper bounds are still very far away from this – we only know $c(n) = O(n^{1-o(1)})$. If we consider a robber of speed s, then the corresponding conjecture is $c_s(n) = O(n^{s/(s+1)})$, which would be best possible [9], but proving this upper bound is completely open.

References

[1] A. Scott, B. Sudakov, A new bound for the cops and robbers problem (2010)

[2] Lu, L. and Peng, X. (2009), On Meyniel's conjecture of the cop number. J. Graph Theory, 71: 192205. doi: 10.1002/jgt.20642

[3] Richard Nowakowski, Peter Winkler, Vertex-to-vertex pursuit in a graph, Discrete Mathematics, Volume 43, Issues 23, 1983, Pages 235-239, ISSN 0012-365X

[4] A. Quilliot, Jeux et pointes fixes sur les graphes, Ph.D thesis, Université de Paris VI, 1978

[5] G. Hahn, Cops, robbers and graphs, Tatra Mt. Math. Publ. 36 (2007), 163176.

[6] A. Bonato, R. J. Nowakowski, The Game of Cops and Robbers on Graphs (2011)

[7] Peter Frankl, Cops and robbers in graphs with large girth and Cayley graphs, Discrete Applied Mathematics, Volume 17, Issue 3, June 1987, Pages 301-305, ISSN 0166-218X

[8] P. Pralat, When does a random graph have constant cop number?, Australasian Journal of Combinatorics 46 (2010) 285-296

[9] Alon, N. & Mehrabian, A. (2011), 'On a Generalization of Meyniel's Conjecture on the Cops and Robbers Game.', Electr. J. Comb. 18 (1) .

[10] Fedor V. Fomin, Petr A. Golovach, Jan Kratochvl, Nicolas Nisse, and Karol Suchan. 2010. Pursuing a fast robber on a graph. Theor. Comput. Sci. 411, 7-9 (February 2010), 1167-1181. DOI=10.1016/j.tcs.2009.12.010 http://dx.doi.org/10.1016/j.tcs.2009.12.010

[11] M. Aigner and M. Fromme, A game of cops and robbers, Discrete Applied Mathematics 8 (1984) 1-12.

[12] N. Alon and J. Spencer, The probabilistic method, 3rd ed., Wiley, Hoboken, NJ, 2008.

[13] Alan Frieze, Michael Krivelevich, Po-Shen Loh, Variations on Cops and Robbers (2010)

[14] P. A. Dreyer, Applications and variations of domination in graphs, Ph.D. Dissertation, Department of Mathematics, Rutgers University, 2000

[15] B. Wieland, A. P. Godbole, On the Domination Number of a Random Graph, The Electronic Journal of Combinatorics, 2001

[16] A. Bonato, G. Hahn, C. Wang, The cop density of a graph, Contributions to Discrete Mathematics, 2007

[17] A. Bonato, P. Pralat, C. Wang, Pursuit-evasion in models of complex networks, Internet Mathematics, 2009

[18] B. Bollobás, G. Kun, I. Leader, Cops and robbers in a random graph, Journal of Combinatorial Theory Series B, 2013

[19] T. Luczak, P. Pralat, Chasing robbers on random graphs: zigzag theorem, Random Structures and Algorithms, 2010

[20] V. Vu, A Large Deviation Result on the Number of Small Subgraphs of a Random Graph. Combinatorics, Probability and Computing, 10, pp 79-94., 2001

[21] F. V. Fomin, P. A. Golovach, J. Kratochvíl, N. Nisse, and K. Suchan, Pursuing a fast robber on a graph, Theor. Comput. Sci. 411 (2010), no. 7-9, 1167-1181.

[22] Bonato, A. and Burgess, A. (2013), Cops and Robbers on Graphs Based on Designs. J. Combin. Designs, 21: 404418. doi: 10.1002/jcd.21331

[23] Frieze, A., Krivelevich, M. and Loh, P.-S. (2012), Variations on cops and robbers. J. Graph Theory, 69: 383402. doi: 10.1002/jgt.20591

[24] F. J. MacWilliams and N. J. A. Sloane, The theory of error correcting codes, North Holland, Amsterdam, 1997.

[25] Chiniforooshan, E. (2008), A better bound for the cop number of general graphs. J. Graph Theory, 58: 4548. doi: 10.1002/jgt.20291

[26] Z. A. Wagner, Cops and Robbers on diameter two graphs, 2014 Preprint